Analytical approximations for a general pension problem

Aleš Ahčan
Kardeljeva ploščad 17, 1000 Ljubljana, Slovenija

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Abstract

In this paper the existing methodology of conditioning Taylor approximation is used to solve a general model from the area of pensions. More specifically, I searched for the optimal multi-period investment strategy of an investor whose accumulation phase (lasting M years) is followed by an annuization period (lasting N years). When choosing the optimal asset mix, I restricted the analysis to the class of constant mix dynamically rebalanced strategies, with optimization criterion set to the probability of default.

I show by means of a numerical illustration that the solutions of the approximate procedure closely relate to the results of Monte Carlo simulation.

As the results indicate increasing the withdrawal rate (annuity) and increasing the investment horizon increases the allocation to risky assets. Namely, when the withdrawal rate gets too large for the risk free investment strategy to provide a 100 % guarantee, the benefits of going more towards risky investment become more and more pronounced.

Keywords: pension problematic, constant mix investment strategy, comonotonic approximations.

1 Introduction

In the past, in contrast to the American system European countries largely supported a more "social" pension system (also known as the "pay as you go" system). The sustainability of this system is based on the sole premise of a constant proportion between the working generation and retirees since the contributions of the active generation are transferred to retirees. Recent demographic trends have raised questions about the long-term stability of such a system and have led to debates about reform of the pension system.

As the examples of some countries indicate part of the solution to this problem can come in the form of increased personal saving. Whether saving is performed through a financial intermediary (i.e mutual fund, bank etc) or directly in the form of investments in the stock and bond market, an important question arises: what is the optimal investment strategy given an investor's age, risk preferences and future consumption?

There is a vast amount of literature on this topic. The first to address this question was Merton (1971), who solved a multi-period portfolio problem of an investor with given consumption and utility defined preferences. Recently, a number of authors (Khorasanee (1996), Milevsky (1998), Milevsky and Robinson (2000), Albrecht and Maurer (2002), Orszag (2002), Gerrard et al. (2003), Dus et al. (2003), Vanduffel et al.(2003), Young (2003) and Dhaene et al. (2005))[1] have focused on a similar problem, which differs from Merton's setting in one important aspect. Within their frameworks the authors calculate the probability that a retiree depletes his wealth while he is still alive (ruin probability or default probability). The approach to defining an investor's objective function through the probability of ruin or the probability of default is in contrast to most of the financial literature on this subject where an investor's preferences are defined via a utility function. Besides the fact that utility functions prove to be a poor representative of an investor's preferences, the approach of optimising the probability of ruin also makes more sense from the point of view of a regulator or financial intermediary, which primarily cares about the stability of their ongoing business.

Despite the enormous theoretical advancements in the area of pensions, there are only a limited number of cases where one can find analytical solutions to proposed models. In most cases, however, solutions can only be obtained through numerical calculations and have little practical value to anyone outside the academics field.

In this paper I wish to remedy this situation by extending the results[2] of Dhaene et al. (2002_a,2002_b, 2003, 2004_b), Hodemakers et al. (2003, 2004_a, 2004_b), Ahčan (2004, $2005_{a,b}$) and Vanduffel et al. (2004, 2005) where the authors develop bounds in the sense of a convex order to yield an approximating sequence for sums of log-normal dependent random variables. In contrast to the work by Dhaene et al. ($2002_{a,b}$) where the authors find approximations for a single signed stream of cash flows I extend their idea of comonotonic approximations to the case of positive and negative cash flows (i.e positive inflows are taken to represent investment, whereas negative cash flows are taken to represent consumption)[3].

Within this theoretical setup I address a general problem from the area of pensions. I search for the optimal investment strategy of an individual who accumulates the wealth necessary for his retirement by periodically investing in a basket of securities for a period of M years. This accumulation phase that lasts M years is then followed by an annuisation period of N years. I search

[1] Josa-Fombellida (2001) searched for the contribution rate (within a DB plan) amortising the unfunded actuarial liability, in order to minimise the contribution rate risk and the solvency risk, while Blake et al. (2003) examined the choices available to a defined contribution (DC) pension plan member at the time of retirement for conversion of his pension fund into a stream of retirement income.

[2] A similar method is found in Ahčan (2005_d).

[3] A similar problem is considered in Vanduffel et al. (2005).

for such an investment strategy that minimises the probability of default (or lifetime ruin probability) given a fixed periodic investment amount during the accumulation phase and a fixed withdrawal rate during the annuisation part.

When choosing the optimal asset mix, I restrict myself to the class of constant mix dynamically rebalanced strategies. As shown by several authors, the constant mix portfolios are in some sense optimal and are proven to be theoretically the most convenient (see Merton (1971), Merton (1990)).

The structure of this paper is as follows. Section 2 introduces the concepts of comonotonicity and stochastic dominance. The next Section gives an overview of the model characteristics: investor preferences, market dynamics. Section 4 explains the main idea behind the approximate approach. The next Section introduces the general pension model. The methodology used to obtain solutions is given in Section 6. The results along with comments are presented in Section 7. In the last Section I give final remarks.

2 Comonotonicity and stochastic dominance

2.1 Comonotonicity, comonotonic sets and comonotonic random vectors

In this subsection I introduce the concepts of comonotonicity, comonotonic sets and comonotonic random vectors. First I state the definition of the comonotonic set (see e.g. Dhaene et al. (2003)).

Let $\overrightarrow{x}, \overrightarrow{y}$ denote two random vectors in R^n and let $\overrightarrow{x} \leq \overrightarrow{y}$ denote componentwise order which is defined by $x_i \leq y_i$ for all $1 \leq i \leq n$.

Definition 1 *The set $A \subseteq R^n$ is said to be comonotonic if for any $\overrightarrow{x}, \overrightarrow{y} \epsilon A$ the following relationship holds $\overrightarrow{x} \leq \overrightarrow{y}$ or $\overrightarrow{y} \leq \overrightarrow{x}$.*

Observe that from the definition it follows that any comonotonic set is simultaneously non-decreasing in each component. Thus a comonotonic set is a thin set, it cannot contain subsets with a dimension bigger than 1. Moreover, any subset of a comonotonic set is also comonotonic.

By having defined a comonotonic set I can proceed to define a comonotonic vector.

Definition 2 *A random vector $\underline{Y} = (Y_1, Y_2, \cdots, Y_n)$ is said to be comonotonic if*

$$(Y_1, Y_2, \cdots, Y_n) \stackrel{d}{=} (F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \cdots, F_{Y_n}^{-1}(U)), \tag{1}$$

where U is a random variable which is uniformly distributed on the unit interval and where the notation $\stackrel{d}{=}$ stands for 'equality in distribution'.
For any random vector $\underline{X} = (X_1, X_2, \cdots, X_n)$, I will call its comonotonic counterpart any random vector with the same marginal distributions and with the comonotonic dependency structure. The comonotonic counterpart of $\underline{X} =$

$(X_1, X_2, \cdots, X_n)$ will be denoted by $\underline{X}^c = (X_1^c, X_2^c, \cdots, X_n^c)$. Hence for any random vector $\underline{X} = (X_1, X_2, \cdots, X_n)$, one has

$$(X_1^c, X_2^c, \cdots, X_n^c) \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \cdots, F_{X_n}^{-1}(U)). \quad (2)$$

It can be proven that a random vector is comonotonic if and only if all its marginals are non-decreasing functions (or all are non-increasing functions) of the same random variable.

2.2 Stochastic dominance

Definition 3 *Consider two random variables X and Y. X is said to precede Y in the stochastic dominance sense, notation $X \leq_{st} Y$, if and only if the distribution function of X always exceeds that of Y:*

$$F_X(x) \geq F_Y(x); \qquad -\infty < x < \infty. \quad (3)$$

Stochastic dominance is a concept often used in actuarial literature to differentiate between more and less "dangerous" random variables. For example, if one has to choose between two risks (losses) X and Y with one preceding the other in a stochastic dominance sense (say $X \leq_{st} Y$) it is trivial to show that all decision-makers prefer X over Y.

3 The model

Consider a problem of an investor who makes periodic investment into the capital market in order to provide himself with a periodic annuity.

3.1 Preferences

The investor chooses his dynamic portfolio so as to minimise the probability of default. Since the default will occur if and only if the end-period wealth is negative (a point that will be made especially clear in one of the following sections), one can choose the optimisation rule with regard to the end period wealth or the probability of negative end-period wealth

$$\min F_{W_n}(0).$$

Here F_{W_n} represents the cumulative distribution function of the end-period wealth W_n, with $F_{W_n}(0)$ giving the probability of default.

Note that the choice of the optimisation criterion contrasts the usual approach within the field of finance where maximisation is done with respect to utility defined preferences. The choice of the probability of default can be justified by the fact that investors and regulators care primarily about the long-term sustainability of their business.

3.2 Market dynamics

In describing the market dynamics I adopt the so-called Black & Scholes framework (see Black et al. 1973).

3.2.1 The Black & Scholes setting

Consider a market of $n+1$ securities which are traded openly and can be bought or sold without incurring any cost. One of the assets is assumed to be risk free, the others are risky. The price of the risk-free asset evolves according to the following deterministic (ordinary) differential equation

$$\frac{\mathrm{d}P^{(0)}(t)}{P^{(0)}(t)} = r\mathrm{d}t,$$

where r stands for the drift or return of the risky asset. Thus the price of the risk-free asset grows exponentially and can be given explicitly by

$$P^{(0)}(t) = P^{(0)} \exp(rt),$$

with $P^{(0)}$ denoting the amount that was invested at time 0.

Other assets are assumed to be risky in the sense that their price is not deterministic and evolves according to a following stochastic differential equation. The price process $P^i(t)$ evolves according to a geometric Brownian motion stochastic process, represented by the following stochastic differential equation:

$$\frac{\mathrm{d}P^i(t)}{P^i(t)} = \mu_i \mathrm{d}t + \sum_{j=1}^{d} \overline{\sigma}_{ij} \mathrm{d}W^j(t), \qquad i = 1, \cdots, m, \tag{4}$$

with $\mu_i > r$ the drift of the i-th risky asset and $\left(W^1(s),\ W^2(s), \cdots,\ W^d(s)\right)$ a d-dimensional standard Brownian motion process. Here it is assumed that the $W^i(s)$ are mutually independent standard Brownian motions.

The diffusion matrix $\overline{\mathbf{\Sigma}}$ is defined by

$$\overline{\mathbf{\Sigma}} = \begin{pmatrix} \overline{\sigma}_{11} & \overline{\sigma}_{12} & \cdots & \overline{\sigma}_{1d} \\ \overline{\sigma}_{21} & \overline{\sigma}_{22} & \cdots & \overline{\sigma}_{2d} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{\sigma}_{m1} & \overline{\sigma}_{m2} & \cdots & \overline{\sigma}_{md} \end{pmatrix} \tag{5}$$

whereas the matrix Σ (reffered to also as the variance-covariance matrix) is defined as

$$\mathbf{\Sigma} = \overline{\mathbf{\Sigma}} \times \overline{\mathbf{\Sigma}}^T = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_m^2 \end{pmatrix}, \tag{6}$$

with coefficients σ_{ij} and σ_i^2 given by $\sigma_{ij} = \sum_{k=1}^{d} \overline{\sigma}_{ik}\, \overline{\sigma}_{jk}$ and $\sigma_i^2 = \sigma_{ii}$. Observe that $\sigma_{ij} = \sigma_{ji}$, hence the matrix is symetric. Additionaly we assume that Σ is positive definite. Thus for all non-zero vectors $\pi^T = (\pi_1,\ \pi_2, \cdots,\ \pi_m)$ we have that all σ_i are strictly positive and that Σ has a matrix inverse Σ^{-1}.

If one defines the process $B^i(s)$ by

$$B^i(s) = \frac{1}{\sigma_i} \sum_{j=1}^{d} \overline{\sigma}_{ij} W^j(s). \tag{7}$$

than equation 4 can be rewritten as:

$$\frac{\mathrm{d}P^i(t)}{P^i(t)} = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}B^i(t), \qquad i = 1, \cdots, m. \tag{8}$$

Observe, that in contrast to equation $W^i(\tau)$ were uncorrelated standard Brownian motion, the $B^i(\tau)$ are (correlated) standard Brownian motions, with

$$Cov\left(B^i(t), B^j(t+s)\right) = \frac{\sigma_{ij}}{\sigma_i\ \sigma_j} t, \qquad t, s \geq 0. \tag{9}$$

The solution to equation (8) is

$$P^{(i)}(t) = P^{(i)} \exp\left((\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i B^{(i)}(t)\right), \tag{10}$$

with $P^{(i)}$ as before denoting the price of i-th risky asset at time 0.

From equation (10) one finds the price of the risky asset to be log-normally distributed with the first two moments given by

$$\mathrm{E}[P^{(i)}(t)] = P^{(i)} \exp\left(\mu_i t\right),$$

$$\mathrm{Var}[P^{(i)}(t)] = (P^{(i)})^2 \exp\left(2\mu_i t\right)\left(\exp(\sigma_i^2 t) - 1\right).$$

A more detailed representation of a multidimensional return process in a Black & Scholes setting can be found in e.g. Björk (1998) or Dhaene et al. (2004_b).

3.2.2 Constant mix investment strategies

In this Section I briefly recapitulate of the most important results on the topic of constant mix investment strategies.

As before, consider a market of n risky and one risk-free security. Within this setting, any investment strategy can be characterised by an allocation vector $\pi(t) = \left(\pi_0(t), \pi_1(t), \ldots, \pi_n(t)\right)^T$, with $\pi_i(t)$ denoting the percentage of the i-th risky asset held at time t, and $\pi_0(t)$ the percentage of risk-free asset in the portfolio. Observe that the fraction placed in the risk-free asset is determined by the aggregate percentage of all risky assets in the portfolio

$$\pi_0(t) = 1 - \sum_{i=1}^{n} \pi_i(t).$$

In the case of a constant mix investment strategy, the percentages (in terms of value) of different assets remain constant over time, so that the time component can be dropped

$$\big(\pi_0(t), \pi_1(t), \ldots, \pi_n(t)\big)^T = \big(\pi_0, \pi_1, \ldots, \pi_n\big)^T.$$

Although the proportions of each asset type are independent of time, the portfolio nevertheless has to be continuously rebalanced in order to keep the percentages of each asset type constant. This strategy implies a "buy low and sell high" principle. Namely, if a price of an asset falls while the prices of all other assets remain constant, one should increase the quantity of that stock (which has fallen) and reduce the quantity of other securities to maintain a constant mix within one's portfolio.

Given a class of constant mix strategies $\vec{\pi}$ one can prove that the portfolio price process $P(t)$ evolves according to the following stochastic differential equation

$$\begin{aligned}\frac{\mathrm{d}P(t)}{P(t)} &= \sum_{i=1}^{m} \pi_i \frac{\mathrm{d}P^i(t)}{P^i(t)} + \left(1 - \sum_{i=1}^{m} \pi_i\right) \frac{\mathrm{d}P^0(t)}{P^0(t)} \\ &= \left(\sum_{i=1}^{m} \pi_i \ (\mu_i - r) + r\right) \mathrm{d}t + \sum_{i=1}^{m} \pi_i \ \sigma_i \mathrm{d}B^i(t).\end{aligned} \tag{11}$$

If we introduce a process $B(\tau)$ by

$$B(\tau) = \frac{1}{\sqrt{\vec{\pi}' \times \mathbf{\Sigma} \times \vec{\pi}}} \sum_{i=1}^{m} \pi_i \ \sigma_i \ B^i(\tau). \tag{12}$$

It can be shown that $B(\tau)$ is a standard Brownian motion, so that we can rewrite equation 11.

$$\frac{\mathrm{d}P(t)}{P(t)} = \mu(\vec{\pi})dt + \sigma(\vec{\pi})\mathrm{d}B(t), \tag{13}$$

with $B(t)$ a standard Brownian motion and $\mu(\vec{\pi})$ and $\sigma^2(\vec{\pi})$ defined as

$$\mu(\vec{\pi}) = r + \vec{\pi}^T \cdot (\vec{\mu} - r\mathbf{1}) \quad \text{and} \quad \sigma^2(\vec{\pi}) = \vec{\pi}^T \cdot \mathbf{\Sigma} \cdot \vec{\pi}.$$

Here $\mathbf{1}$ denotes the m-dimensional vector of ones $(1, 1, \ldots, 1)^T$, and $\mathbf{\Sigma}$ stands for a variance-covariance matrix which is assumed to be positive definite. The solution of equation (13) is

$$P(t) = P \exp\big((\mu(\vec{\pi}) - \frac{1}{2}\sigma^2(\vec{\pi}))t + \sigma(\vec{\pi})B(t)\big), \tag{14}$$

with expectation and variance given by

$$\mathrm{E}[P(t)] = P \exp\big(\mu(\vec{\pi})t\big),$$

$$\mathrm{Var}[P(t)] = P^2 \exp\big(2\mu(\vec{\pi})t\big)\big(\exp(\sigma^2(\vec{\pi})t) - 1\big).$$

Throughout this Section I use the concept of yearly return, which gives the log-value of one money unit investment after a one-year period. In line with equation (14) a return in year k can be written as

$$Y_k(\vec{\pi}) = \mu'(\vec{\pi}) + \sigma(\vec{\pi})\big(B(k) - B(k-1)\big),$$

with $\mu'(\vec{\pi})$ (where $\mu'(\vec{\pi})$ is equal to $\mu(\vec{\pi}) - \frac{1}{2}\sigma^2(\vec{\pi})$) denoting the drift, $\sigma(\vec{\pi})$ standard deviation (on a yearly basis) of investment strategy $\vec{\pi}$ and $B(k)$ stands for a standardised Brownian motion. In more general terms, the value of a single unit investment over a period of k years expressed in terms of yearly returns can be written as

$$P(k) = P\exp\big(Y_1(\vec{\pi}) + Y_2(\vec{\pi}) + \cdots + Y_k(\vec{\pi})\big).$$

Observe that the yearly returns $Y_1(\vec{\pi})$ are independent and normally distributed, hence the return over a period of k years is also normally distributed.

4 Conditioning on the first order Taylor series expansion

In this Section I give a brief overview of the basics of the approximating procedure of conditioning Taylor approximation. A more detailed explanation of the topic can be found in papers by Dhaene et al. ($2002_{a,b}$) and Ahčan ($2005_{a,c}$).

As I will be dealing with the sums/differences of lognormal random variables it makes sense to explain the methodology in a similar case. Thus, consider a linear combination of lognormal variables

$$L_n = \sum_{i=1}^{n} \alpha_i \exp\big(Y(i)\big), \tag{15}$$

with α_i denoting payments made in year i and $Y(i)$ denoting the return on investment made in year i (more precisely $Y(i)$ denotes the return from year i to year n).

Observe that $\overrightarrow{Y} = (Y(1), Y(2), \cdots, Y(n))$ is a random vector following the multivariate normal law and we thus need to find an approximating sequence for a sum of log-normally distributed dependent random variables.

In general (if at least two of yearly payments α_i are non-zero and $n \geq 2$) it is not possible to obtain an analytical expression for the distribution function of L_n. When one tries to solve this problem two quick solutions come to mind. An obvious and well-practiced approach involves using a Monte Carlo simulation, which is useful to the extent to which it can be made accurate. Unfortunately, this comes at a price: the time required to perform such simulations is often unrealistic. One way to avoid time-consuming Monte Carlo simulations is to approximate the sum in question by a first-order Taylor expression which yields an approximating sequence that is normally distributed. Yet a convenient solution that allows one to easily calculate the distribution function and its quantiles is at the same time very limited. Namely, this approach will work well only if the variances of the components of the vector of returns $\overrightarrow{Y}$ are very small, which in practice will hardly ever be the case.

Under the time constraint of using the Monte Carlo simulation and the limited applicability of first-order Taylor approximation, one may ask if there is a method that is at the same time sufficiently accurate and not time consuming.

As first shown by Rogers and Shi (1995) the technique of conditioning provides such a solution since it is both highly accurate and analytical. Following their methodology, the sum in question (15) can be approximated by the following sequence

$$E\left[L_n \mid \Lambda\right] = \sum_{i=1}^{n} \alpha_i \cdot E\left[e^{Y(i)} \mid \Lambda\right], \tag{16}$$

where E denotes the expectation operator with the conditioning variable[4] Λ defined by

$$\Lambda = \sum_{i=1}^{n} \alpha_i \cdot e^{E[Y(i)]} \cdot Y(i). \tag{17}$$

Before explaining the logic behind this approach, first observe that the conditioning random variable is equal to the first-order Taylor expansion of (15) around the expected values of $Y(i)$ (the only difference between the two is in the constant term which in no way influences the statistical properties of the approximated sum).

Now I may proceed to explain why the technique of conditioning Taylor is superior to the first-order Taylor series approximation. Note that the first-order Taylor series expansion misinterprets the original sum or, more accurately, the terms which appear in the approximated sum; a sum of log-normals is replaced by a sum of normally distributed variables with each log-normally distributed term (random variables) being replaced by its normally distributed counterparts (in terms of the distribution function). In contrast to the first-order Taylor series expansion, the conditioning technique gives both an accurate description of the stochastic process by accurately capturing the statistics of the return vector with the choice of the conditioning random variable Λ (i.e note that the set of all outcomes of L_n is well determined by the set of (events or) possible realisations of the return vector $\overrightarrow{Y}$) and, at the same time, adequately describes (approximates) the terms appearing in the sum; a log-normal term is replaced by its log-normally distributed counterpart. Thus the conditioning technique can be regarded as a two level process; at the first level the statistics of the process L_n(randomness) are captured by the conditioning random variable Λ, and at the second level the sum of returns (defined by the set of outcomes in Λ) is transformed by the technique of conditioning to the sum of exponents of

[4] For more on choosing the appropriate conditioning random variable, see Vanduffel et al. (2004).

returns (log-normal random variables). As shown in Dhaene et al. ($2002_{a,b}$), the choice of conditioning random variable in (17) yields an approximating sequence

$$E\left[L_n \mid \Lambda\right] = \sum_{i=0}^{n} \alpha_i \; e^{(n-i)\;\mu(\boldsymbol{\pi})+\frac{1}{2}(1-r_i^2(\boldsymbol{\pi}))\;(n-i)\;\sigma^2(\boldsymbol{\pi})+r_i(\boldsymbol{\pi})\sqrt{n-i}\;\sigma(\boldsymbol{\pi})\;\Phi^{-1}(U)}, \tag{18}$$

where U is an uniformly distributed random variable, that follows from

$$\Phi^{-1}(U) = \frac{\Lambda - E(\Lambda)}{\sigma_\Lambda} \tag{19}$$

and r_i is the correlation coefficient between $Y(i)$ and Λ

$$r_i\left(\boldsymbol{\pi}\right) = \frac{\sum_{j=i+1}^{n}\sum_{k=0}^{j-1}\alpha_k\;e^{k\;\mu(\boldsymbol{\pi})}}{\sqrt{n-i}\sqrt{\sum_{j=1}^{n}\left(\sum_{k=0}^{j-1}\alpha_k\;e^{k\;\mu(\boldsymbol{\pi})}\right)^2}}. \tag{20}$$

Observe that the correlation coefficients $r_i\left(\boldsymbol{\pi}\right)$ are non-negative, which implies that the sum in (18) is strictly increasing in $\Phi^{-1}(U)$. Therefore, the quantiles of the distribution function in (18) are equal to

$$Q_p(E\left[L_n \mid \Lambda\right]) = \sum_{i=0}^{n} \alpha_i \, e^{(n-i)\;\mu(\boldsymbol{\pi})+\frac{1}{2}(1-r_i^2(\boldsymbol{\pi}))\;(n-i)\;\sigma^2(\boldsymbol{\pi})+r_i(\boldsymbol{\pi})\;\sqrt{n-i}\;\sigma(\boldsymbol{\pi})\;\Phi^{-1}(p)}, \tag{21}$$

with Q_p denoting the quantile of the distribution function of $E\left[L_n \mid \Lambda\right]$.

In contrast to the first-order Taylor series expansion, the conditioning technique has a much wider range of applicability; as long as the volatility of returns is below 30% the approximations in (21) provide an excellent fit against simulated quantiles (see, for example, Dhaene et al. ($2002_{a,b}$), Milevsky (2004), Ahčan et al. (2004, 2005_a)).

5 A general pension problem

As mentioned we consider a multi-period portfolio problem of an investor who has to accumulate enough wealth by periodically investing in the capital market to fulfill a set of future deterministic obligations (annuities). The investor's horizon consists of an investment phase that is taken to last M years and an annuisation period of N years.

The amounts α_i (deposits) are assumed to be non-negative, with $i = 1, ., M$, while β_i (i.e. withdrawals, consumption or annuities) are assumed to be non-positive, with $i = M+1, ., M+N$. As depicted in the picture the end-period wealth $S(M+N)$ is then simply equal to

$$S(M+N) = \sum_{i=1}^{M+N} \alpha_i \cdot e^{Z(i)} - \sum_{i=M+1}^{M+N} \beta_i \cdot e^{Z(i)}, \tag{22}$$

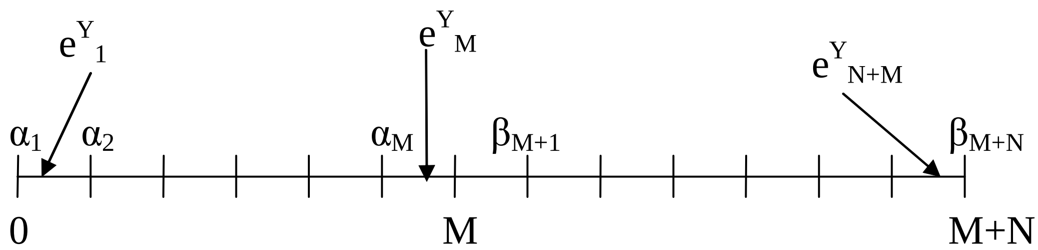

Figure 1: Multi-period consumption and savings.

with $Z(i)$ denoting the returns over the investment horizon

$$Z(i) = (Y_i + ... + Y_{M+N}); i \leq M, \tag{23}$$
$$Z(i) = (Y_{i+1} + ... + Y_{M+N}); i > M. \tag{24}$$

Observe that the first term in (22) gives the accumulated value of investments at the end of the investment horizon while the last part gives the sum of consumption over the period (discounted to the end of the investment horizon). Thus $S(M+N)$ can be regarded as the future value of the difference between the two (investment minus consumption).

One can quite easily show that the probability of default over the whole investment period (a default is declared if the amount of wealth at any time $t \leq T$ is smaller than the consumption β_i over the same period) is simply equal to the probability of negative end-period wealth

$$\Pr(S(t) \leq 0; M \leq t \leq M+N) = \Pr(S(M+N) \leq 0). \tag{25}$$

This result is a simple consequence of the fact that if inter-temporal wealth (the difference between the accumulated wealth and consumption over that period) is negative at any point in time, it will also be negative at the end. Namely, consumption only reduces wealth (if wealth before consumption is negative it will remain negative thereafter), while the return process just enriches what one had in the previous period (investment return only has an influence on the absolute amount of wealth). Due to that, equation (25) can also be restated in another form. Consider a random variable $S(M)$, defined as

$$S(M) = S(M+N) \cdot e^{-(Y_{M+1}+...+Y_{M+N})} = \sum_{i=1}^{M} \alpha_i \cdot e^{G(i)} - \sum_{i=M+1}^{M+N} \beta_i \cdot e^{-G(i)}, \tag{26}$$

with $G(i)$ defined as

$$G(i) = (Y_i + ... + Y_M); i \leq M, \tag{27}$$
$$G(i) = (Y_{M+1} + ... + Y_i); i > M. \tag{28}$$

In simple terms $S(M)$ gives the discounted value of the end-period wealth $S(M+N)$, where discounting is done with respect to the intermediate point M. From equation (26) it is not difficult to see that the probability of default of our transformed random variable $S(M)$ is equal to the default probability of $S(M+N)$; the result being obvious since a negative outcome will always be transformed into a negative value

$$\Pr(S(M) \leq 0) = \Pr(S(M+N) \leq 0). \tag{29}$$

Note that the transformed random $S(M)$ has more desirable properties in terms of dependency structure than $S(M+N)$. In contrast to $S(M+N)$ both parts of $S(M)$ (the investment and the consumption part) are independent of each other. Since (26) is the difference of two independent parts, with each part being a sum of log-normally distributed dependent random variables, one can successfully deploy the methodology presented in the previous Section to yield an accurate comonotonic approximating sequence.

6 Solution methodology

6.1 Approximating procedure

In this Section I rely on the results of Section 4 to find an approximate analytical expression for the distribution function of $S(M)$

$$S(M) = \sum_{i=1}^{M} \alpha_i \cdot e^{G(i)} - \sum_{i=M+1}^{M+N} \beta_i \cdot e^{-G(i)}. \tag{30}$$

In deriving the conditioning random variable Λ I use the first-order Taylor expansion of expression (30)

$$\Lambda = \sum_{i=1}^{M} \alpha_i \cdot e^{E[G_i]} \cdot G(i) + \sum_{i=M+1}^{M+N} \beta_i \cdot e^{-E[G_i]} \cdot G(i), \tag{31}$$

where the equation (31) can also be rewritten in a more general form

$$\Lambda = \sum_{i=1}^{M+N} \gamma_i \cdot Y_i\left(\boldsymbol{\pi}\right), \tag{32}$$

with γ_i equal to[5]

$$\begin{aligned} \gamma_i &= \sum_{k=1}^{M} \alpha_k \cdot e^{\mu(\boldsymbol{\pi})\cdot k};\ \text{if } i \leq M, \\ \gamma_i &= \sum_{k=M+1}^{M+N} \beta_k \cdot e^{-\mu(\boldsymbol{\pi})\cdot (k-M)};\ \text{if } i > M. \end{aligned} \tag{33}$$

[5] In determing the coefficients a similar approach has been used as in Dhaene et al. (2004_b)

With this choice of conditioning random variable the conditioning Taylor approximating sequence can be expressed as[6]

$$E(S(M) \mid \Lambda) = \sum_{i=1}^{M} \alpha_i \cdot \; e^{i\;\left(\mu(\boldsymbol{\pi})-\frac{1}{2}r_i^2(\boldsymbol{\pi})\;\sigma^2(\boldsymbol{\pi})\right)+r_i(\boldsymbol{\pi})\;\sqrt{i}\;\sigma(\boldsymbol{\pi})\;\Phi^{-1}(U)} - \tag{34}$$
$$- \sum_{i=M+1}^{M+N} \beta_i \cdot e^{-(i-\;M)(\mu(\boldsymbol{\pi})+(1-\frac{1}{2}r_i^2(\boldsymbol{\pi})))\;\sigma^2(\boldsymbol{\pi})-r_i(\boldsymbol{\pi})\;\sqrt{(i-\;M)}\;\sigma(\boldsymbol{\pi})\;\Phi^{-1}(U)},$$

with r_i given by

$$\begin{aligned} r_i &= \frac{\sum_{k=i}^{M}\gamma_k}{\sqrt{i}\cdot\sqrt{\sum_{i=1}^{M+N}\gamma_i^2}};\text{ if } i \le M, \\ r_i &= \frac{\sum_{k=M+1}^{i}\gamma_k}{\sqrt{(i-M)}\cdot\sqrt{\sum_{i=1}^{M+N}\gamma_i^2}};\text{ if } i > M \end{aligned} \tag{35}$$

and γ_i defined as in (33).

Note that due to the fact that all r_i are positive (or non-negative) the difference of sums in (34) is strictly increasing in $\Phi^{-1}(U)$ and thus comonotonic. Hence, the quantiles can be given by

$$Q_p(E(S(M) \mid \Lambda)) = \sum_{i=1}^{M} \alpha_i \cdot \; e^{i\;\left(\mu(\boldsymbol{\pi})-\frac{1}{2}r_i^2(\boldsymbol{\pi})\;\sigma^2(\boldsymbol{\pi})\right)+r_i(\boldsymbol{\pi})\;\sqrt{i}\;\sigma(\boldsymbol{\pi})\;\Phi^{-1}(p)} - \tag{36}$$
$$- \sum_{i=M+1}^{M+N} \beta_i \cdot e^{-(i-\;M)(\mu(\boldsymbol{\pi})+(1-\frac{1}{2}r_i^2(\boldsymbol{\pi})))\;\sigma^2(\boldsymbol{\pi})-r_i(\boldsymbol{\pi})\;\sqrt{(i-\;M)}\;\sigma(\boldsymbol{\pi})\;\Phi^{-1}(p)}.$$

Recall, again that the goal when selecting the optimal asset allocation is to minimise the default probability over the whole investment period, which is equal to the default probability of the transformed random variable $S(M)$. The optimality is thus achieved for a strategy that minimises the probability of negative $S(M)$

$$p = \min_{\pi} F_{S(M)}(0). \tag{37}$$

Observe that no analytical expression exists for either the distribution function or the quantiles of $S(M)$. Thus I approximate the default probability of $S(M)$ with the default probability of the conditioning Taylor sequence $E(S(M) \mid \Lambda)$

$$p^{app} = \min_{\pi} F_{E(S(M)|\Lambda)}(0). \tag{38}$$

[6] Exact form of the approximating sequence is obtained by considering $E\left[Y_i\right] = \mu_i - \frac{1}{2}\sigma_i$

The default probability in (38) can be obtained by solving the following equation

$$\sum_{i=1}^{M} \alpha_i \cdot \; e^{i\;\left(\mu(\boldsymbol{\pi}) - \frac{1}{2} r_i^2(\boldsymbol{\pi})\;\sigma^2(\boldsymbol{\pi})\right) + r_i(\boldsymbol{\pi})\;\sqrt{i}\;\sigma(\boldsymbol{\pi})\;\Phi^{-1}(p)} - \tag{39}$$

$$- \sum_{i=M+1}^{M+N} \beta_i \cdot e^{-(i-\;M)(\mu(\boldsymbol{\pi}) + (1 - \frac{1}{2} r_i^2(\boldsymbol{\pi})))\;\sigma^2(\boldsymbol{\pi}) - r_i(\boldsymbol{\pi})\;\sqrt{(i-\;M)}\;\sigma(\boldsymbol{\pi})\;\Phi^{-1}(p)} = 0.$$

Since (39) is increasing in $\Phi^{-1}(p)$ the solution to the equation is unique and the exact value can be obtained by using one of the iterative methods like Newton-Raphson.

6.2 Restricting the set of solutions

In the previous Section I showed how conditioning Taylor approximation can be deployed to yield an analytical description of the quantiles of the distribution function of $S(M)$. Although this partially solves the problem of multidimensionality, it still does not allow one to obtain a solution in realistic time. A larger part of the multidimensionality[7] curse is namely related to numerous assets which result in an almost "infinite" number of combinations one has to consider when searching for the optimal asset mix. Thus it makes sense to prove that the solution to the problem (37) can be found in a subset of the set of all possible solutions.

Recall that the efficient frontier is defined as the solution to the following minimisation problem (see Markowitz (1952))

$$\min_{\vec{\pi}} \sigma(\vec{\pi}) \quad \text{subject to} \quad \mu(\vec{\pi}) = \mu. \tag{40}$$

Alternatively, the efficient frontier can also be defined as a maximisation problem by interchanging the variables in equation (40)

$$\max_{\vec{\pi}} \mu(\vec{\pi}) \quad \text{subject to} \quad \sigma(\vec{\pi}) = \sigma. \tag{41}$$

It is not hard to show (by means of a Lagrange optimisation) that under the assumption of positive definite variance-covariance matrix $\boldsymbol{\Sigma}$ and allowed short selling a unique solutions to the problem in (41) can be obtained

$$\mu(\vec{\pi}_\sigma) = r + \sigma\sqrt{(\vec{\mu} - r\mathbf{1})^T \cdot \boldsymbol{\Sigma}^{-1} \cdot (\vec{\mu} - r\mathbf{1})} \tag{42}$$

and

$$\vec{\pi}_\sigma = \sigma \frac{\boldsymbol{\Sigma}^{-1} \cdot (\vec{\mu} - r\mathbf{1})}{(\vec{\mu} - r\mathbf{1}) \cdot \boldsymbol{\Sigma}^{-1} \cdot (\vec{\mu} - r\mathbf{1})}, \tag{43}$$

[7] The multidimensionality curse relates to the problem of numerous combinations of possible portfolios of assets one has to consider and the problem of determination of the distribution function of a sum of dependent log-normal random variables.

where $\vec{\pi}_\sigma$ corresponds to the optimal investment strategy giving rise to the maximum in (41).

Observe that (43) can be rewritten as

$$\vec{\pi}_\sigma = \frac{\mu(\vec{\pi}_\sigma) - r}{\mu(\vec{\pi}_t) - r} \cdot \vec{\pi}_t, \tag{44}$$

where $\vec{\pi}_t$ denotes the tangency portfolio. Thus the mean-variance optimizing investors differ only with respect to the percentage of assets they place in the tangency portfolio[8].

Theorem 4 *The solution to the problem (37) is to be found on the Capital Market Line (CML).*

Proof. Observe that the return over an investment horizon of i periods denoted $Z(i)$ can be expressed in terms of yearly returns $Y_i(\vec{\pi})$

$$Z(i) = \sum_{k=1}^{i} Y_k. \tag{45}$$

Since each yearly return can be written as

$$Y_k = \mu + \sigma\Phi^{-1}(U_k),$$

with U_i an independent $(0,1)$ uniform random variable, I can rewrite equation (45) to get

$$Z(i) = i\mu + \sigma\sum_{k=1}^{i} \Phi^{-1}(U_k).$$

It is easy to show that if one wants to maximise a return over any investment horizon it is optimal to choose a portfolio from CML. For any given σ the portfolios from CML will have the highest expected drift μ and will thus stochastically dominate all other portfolio strategies

$$\underset{\vec{\pi}\epsilon CML}{Z_i} \geq_{st} \underset{\vec{\pi}\notin CML}{Z_i}.$$

Therefore it is also optimal to select an investment strategy from CML if one wants to maximise the future value of a series of periodic investments α_i

$$S^+ = \sum_{i=1}^{N} \alpha_i e^{i\mu} e^{\sigma\Phi^{-1}(U_i)}$$

or to minimise the present value of a series of periodic withdrawals β_i

$$S^- = \sum_{i=1}^{N} \beta_i e^{-i\mu} e^{\sigma\Phi^{-1}(V_i)}.$$

[8] See Dhaene et. al (2005) for details.

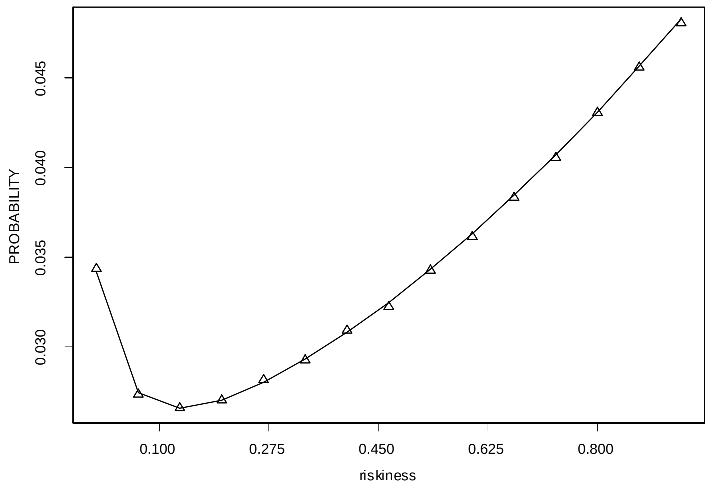

Figure 2: Default probability as a function of investment strategy ($\beta = 1.3$).

Since $S(M)$ is just the difference of the future wealth earned by periodic investments and the present value of future obligations

$$S(M) = S^{+} - S^{-} = \sum_{i=1}^{M} \alpha_i \cdot e^{G(i)} - \sum_{i=M+1}^{M+N} \beta_i \cdot e^{-G(i)}, \tag{46}$$

our result follows. ■

7 Numerical illustration

In this Section I explicitly work out the general pension problem discussed in Section 5. By comparing the results of the conditioning Taylor approximation and Monte Carlo simulation I test the quality of my approximate approach. For the purpose of numerical illustration the following set of parameters[9] was chosen: $M = N = 20$, $\alpha_1 = ... = \alpha_{20} = 1, \beta_1 = ... = \beta_{20} = -1.3$, risk-free rate $r_f = 0.014$, drift of the stock market index $\mu_m = 0.073$ and standard deviation $\sigma_m = 0.16$.

Figure 2 presents the relationship between the probability of default (y axis) and the investment strategy (x axis). The investment strategy is characterised

[9] A similiar choice of parameters can be found in Campbell et al. (2003) and Dhaene (2004_b).

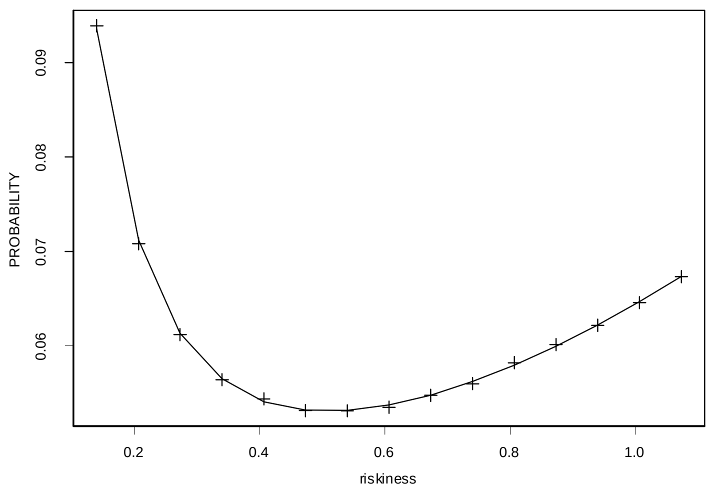

Figure 3: Default probability as a function of investment strategy ($\beta = 1.4$).

by an allocation pattern between risky and risk-free assets (in the figure the percentage of investment in a market index is given). As one can see from the figure, the conditioning Taylor approach (solid line) gives an excellent fit against the simulated probability of default (solid triangles). The maximum relative difference between the default probability calculated from conditioning Taylor and the default probability from the MC simulation is 1.2%. As the figure illustrates, the deviation between the simulated and approximated results is in no way restrictive; both the optimal investment strategy and the level of default probability almost coincide for the two procedures. The optimal investment strategy (characterised by an allocation between the risky and risk-free investment, where the second value represents the fraction of wealth invested in risky assets) obtained by means of the conditioning Taylor approximation is equal to $\pi_{app} = (86.6\%; 13.4\%)^T$ with the corresponding default probability $F_{app}(0) = 0.0262$. These values closely resemble those obtained from simulation $\pi_{MC} = (86.6\%; 13.4\%)^T$ with $F_{MC}(0) = 0.0265$.

In the second numerical example, I increase the withdrawal rate β (-1.4 instead of -1.3) in order to examine the effect of a change on the optimal asset mix and the default probability. As one can expect the default probability increases; in the case of the conditioning Taylor approximation the default probability is equal to $F_{app}(0) = 0.052$ whereas the simulated value is equal to $F_{MC}(0) = 0.0525$. This should be measured against the shift in the optimal investment strategy $\pi_{app} = (50.6\%; 49.4\%)^T$ for the conditioning Taylor approximation versus $\pi_{MC} = (50\%; 50\%)^T$ in the case of the Monte Carlo simulation.

Again, the deviation between the simulated and the approximated values is relatively small (1%) and in no way influences the choice of the optimal investment strategy.

The results of both numerical illustrations indicate that changing the withdrawal rate influences both the optimal investment strategy as well as the default probability. The default probability increases with the annuity, whereas the asset mix becomes riskier with increasing annuity. The resulting shift of an investment strategy leaning towards risky assets is at all not surprising and can be well explained by considering the interplay between default probability and the amount of obligations (annuities). Namely, when the annuity level gets too large for the risk-free investment strategy to provide a 100% guarantee, the benefits of going more towards risky investments become more and more pronounced. Note that once the annuity is larger than the border annuity, which is defined as the largest amount of withdrawals for which the default probability is still zero, then the default probability of a risk-free investment will be different from zero, actually 100% (the result of the risk-free investment strategy is always dichotomous) and the benefits of riskier strategies become more pronounced. With increasing annuities the minimal default probability inherently increases and the optimum can be achieved for strategies with increasingly higher proportions of risky assets.

8 Conclusion

I have searched for the optimal multi-period investment strategy of an investor whose accumulation phase lasting M years is followed by an annuisation/consumption period of N years. When choosing the optimal asset mix, I restricted the analysis to the class of constant mix dynamically rebalanced strategies, with the optimisation criterion set to the probability of default.

In solving the model I rely on the technique of conditioning Taylor approximation. I have shown by means of a numerical illustration that the solutions of my approximate procedure closely relate to the results of a Monte Carlo simulation, with the maximum relative deviation between the approximated results and those obtained by Monte Carlo being negligible (1%).

As the results indicate, increasing the withdrawal rate (annuity) and increasing the investment horizon increases the allocation to risky assets. Namely, when the withdrawal rate becomes too large for the risk-free investment strategy to provide a 100% guarantee, the benefits of moving more towards a risky investment become more and more pronounced.

My work can lead to several possible generalisations. Perhaps the most important obstacle when analysing the results of this model is the problem of infinite transaction cost due to continuous rebalancing. Although limiting in some sense, this problem can be well accounted for with a slight modification of the existing methodology (Ahčan (2005_b)). A less serious problem is linked to the assumption of log-normal returns. As I model market yearly returns this assumption is not far from reality. But if one were to model single-asset returns (or index returns over shorter time horizons) then the assumption of log-normal returns is no longer valid and one is bound to consider other alternatives, such as stable non-Gaussian distribution functions. Another possible extension linked to the assumption of independent yearly returns is to include some sort of dependence; as some authors indicate the yearly returns show slight dependence in the form of a mean reversion (see Fama and French (1988), Bessembinder et al. (1995), Balvers et al. (2000), Campbell et al. (2003), Munk et al. (2004) and Gropp (2004) for more details).

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